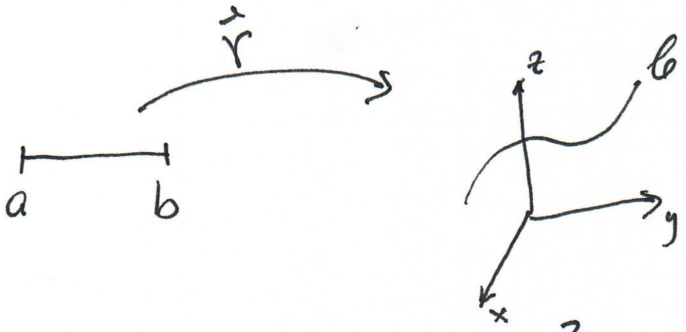
16.6 Parametric Surfaces and Their Areas.

Curves in 3-D are represented in parametric form as

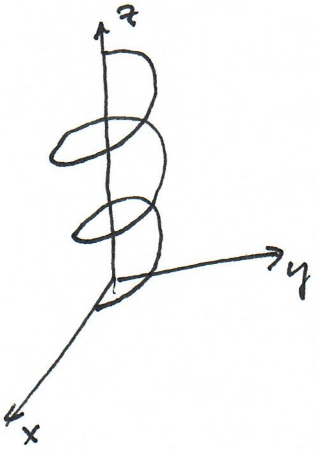
$$\mathcal{C}: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle \quad , \quad a \le t \le b$$

$$\text{or} \quad x = x(t), \quad y = y(t), \quad z = z(t)$$



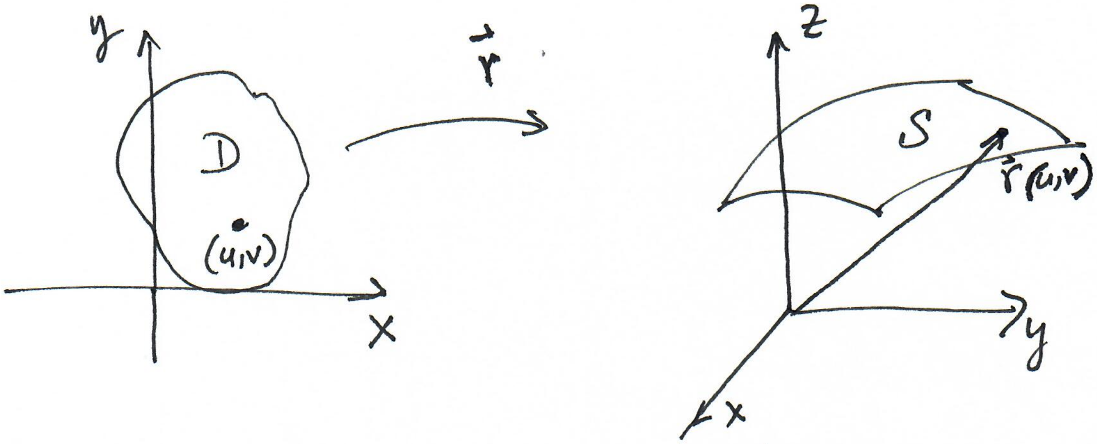
$$\vec{r} : [a,b] \longrightarrow \mathbb{R}^3$$

Example: $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 3\pi$



Similarly, Surfaces can be represented parametrically by

$$\vec{r}: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$



$$\vec{r}(u,v) = \left\langle x(u,v), y(u,v), z(u,v) \right\rangle, \quad (u,v) \in D \subseteq \mathbb{R}^2$$

16.6

Parametric Surfaces and Their Areas.

Example 1.- **Cylinder**

$x^2+y^2=9$,

$0\le z\le 10$

parametrization:

$$\left.\begin{aligned} x &= x(\theta,z) = 3\cos\theta \\ y &= y(\theta,z) = 3\sin\theta \\ z &= z(\theta,z) = z \end{aligned}\right\} \quad \begin{aligned} 0\le\theta\le 2\pi \\ 0\le z\le 10 \end{aligned}$$

or $\vec{r}(\theta,z) = \langle 3\cos\theta, 3\sin\theta, z\rangle$

Example 2.- **Sphere**

$x^2+y^2+z^2=a^2$

$$\left\{\begin{aligned} x &= x(\phi,\theta) = a\sin\phi\cos\theta, \\ y &= y(\phi,\theta) = a\sin\phi\sin\theta, \\ z &= z(\phi,\theta) = a\cos\phi, \end{aligned}\right. \quad \begin{aligned} 0\le\phi\le\pi \\ 0\le\theta\le 2\pi \end{aligned}$$

If $u=\phi$ and $v=\theta$

$\Rightarrow \vec{r}(u,v) = \langle a\sin u\cos v, a\sin u\sin v, a\cos u\rangle$ $\quad 0\le u\le\pi$, $0\le v\le 2\pi$

Example 3.- **Paraboloid** : $z=x^2+y^2$, $\quad 0\le z\le 4$

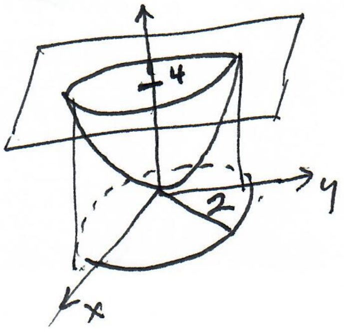
I) $\vec{r}(x,y) = \langle x, y, x^2+y^2\rangle$ or $\vec{r}(u,v) = \langle u, v, u^2+v^2\rangle$

$-2\le u\le 2$, $-\sqrt{4-u^2}\le v\le \sqrt{4-u^2}$

or II) $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2\rangle$

$0\le\theta\le 2\pi$

$0\le r\le 2$



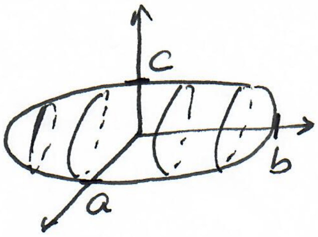
Example 4 $r(u,v) = \langle (2+\sin v)\cos u, (2+\sin v)\sin u, u+\cos v \rangle$

$0 \le u \le 4\pi$
$0 \le v \le 2\pi$

(Example 2 in book 7E).

Example 5 Ellipsoid with semi-axes $a, b,$ and c

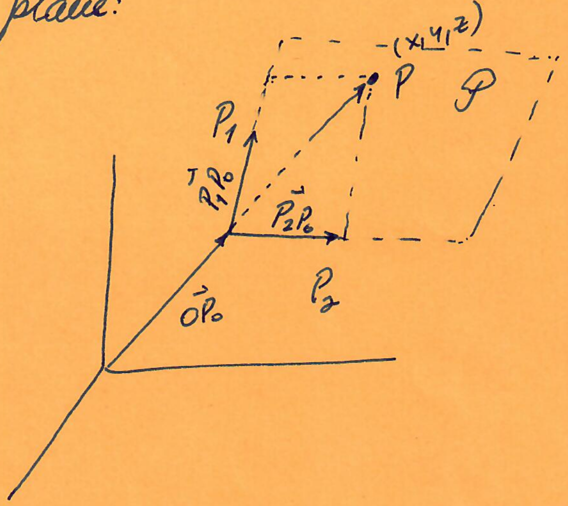
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



$$S: \begin{cases} x = a\cos u \sin v \\ y = b \sin u \sin v \\ z = c\cos v \end{cases} \qquad \begin{array}{l} u \in [0, 2\pi] \\ v \in [0, \pi] \end{array}$$

b2) Another parametrization of the plane:

$$\boxed{\mathcal{P}: \quad 3x + 2y - 2z = 5}$$



Consider the three points:

$P_0(1,1,0)$, $P_1(3,-1,1)$, $P_2(5,-1,4)$

on the plane $\mathcal{P}$

and form the vectors:

$$\vec{P_1P_0} = \vec{\omega}_1 = (3,-1,1) - (1,1,0) = (2,-2,1)$$

$$\vec{P_2P_0} = \vec{\omega}_2 = (5,-1,4) - (1,1,0) = (4,-2,4)$$

then any point $P(x,y,z)$ in $\mathcal{P}$ satisfies

$$\vec{PP_0} = u\vec{\omega}_1 + v\vec{\omega}_2 + \vec{OP_0}$$

$$= u(2,-2,1) + v(4,-2,4) + (1,1,0).$$

$$= (2u + 4v + 1, -2u - 2v + 1, u + 4v)$$

or parametric representation of plane $\mathcal{P}$.

$$\begin{cases} x = x(u,v) = 2u + 4v + 1 \\ y = y(u,v) = -2u - 2v + 1. \\ z = z(u,v) = u + 4v \end{cases}$$

Parametric Representation of cone: (Half-cone)

$$z^2 = 3(x^2+y^2), \quad z \geq 0$$

Using spherical coordinates

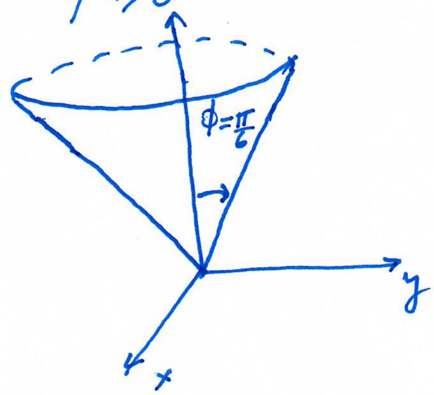
$$\begin{cases} x = \rho \sin\phi \cos\theta, & 0 \leq \theta \leq 2\pi \\ y = \rho \sin\phi \sin\theta, & 0 \leq \phi \leq \pi \\ z = \rho \cos\phi \end{cases}$$

$$\rho^2 \cos^2\phi = 3\rho^2 \sin^2\phi \Rightarrow \tan^2\phi = \frac{1}{3} \Rightarrow \tan\phi = \frac{\sqrt{3}}{3}$$

$$\Rightarrow \begin{matrix} \sin\phi = 1/2 \\ \cos\phi = \sqrt{3}/2 \end{matrix} \Rightarrow \phi = \frac{\pi}{6}$$

Then, a natural parametrization is

$$\begin{cases} x = \rho \sin\pi/6 \cos\theta = \frac{\rho}{2}\cos\theta, & 0 \leq \theta \leq \pi \\ y = \rho \sin\pi/6 \sin\theta = \frac{\rho}{2}\sin\theta, & \rho \geq 0 \\ z = \rho \cos\pi/6 = \frac{\sqrt{3}}{2}\rho \end{cases}$$



If we call $u = \rho$ and $v = \theta$

Then,

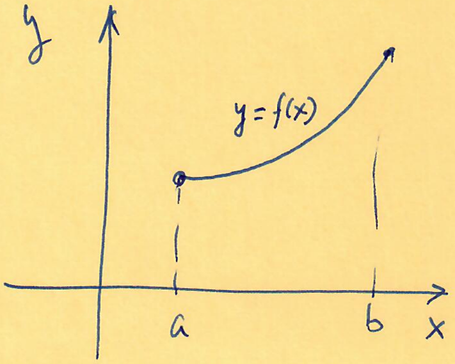
$$\begin{cases} x = \frac{1}{2} u \cos v, & u \geq 0 \\ y = \frac{1}{2} u \sin v, & 0 \leq v \leq 2\pi. \\ z = \frac{\sqrt{3}}{2} u \end{cases}$$

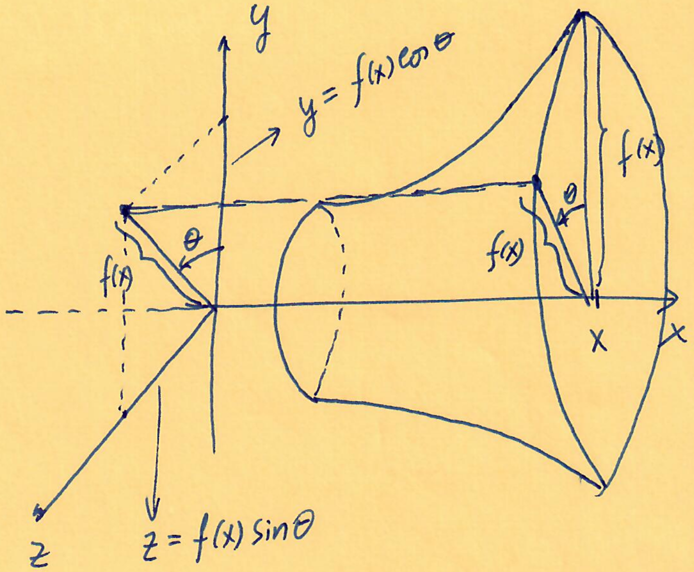
PS$_x$

f) Surfaces of revolution

Consider the function $y = f(x)$, $a \le x \le b$.

And rotate it about the x-axis.



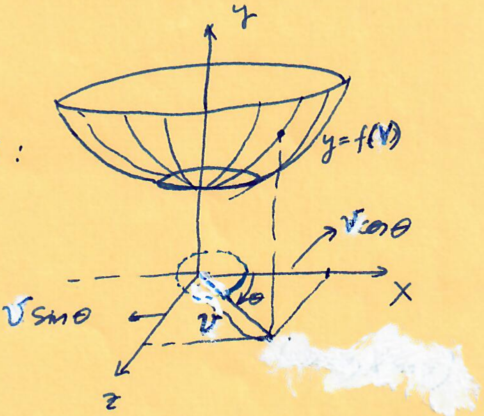


Parametric equations: $\theta = v$

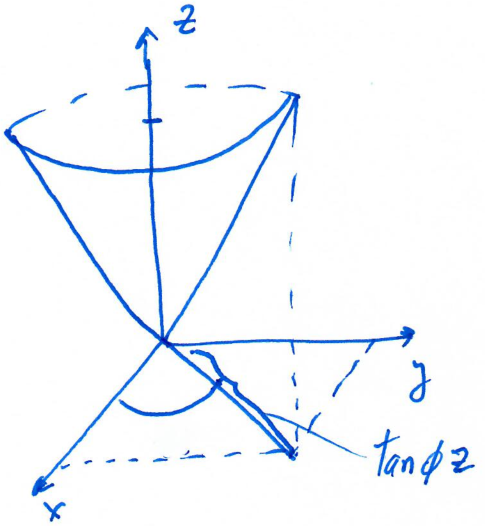
$$\begin{cases} x = u & a \le u \le b \\ y = f(u)\cos v & 0 \le v \le 2\pi. \\ z = f(u)\sin v & \end{cases}$$

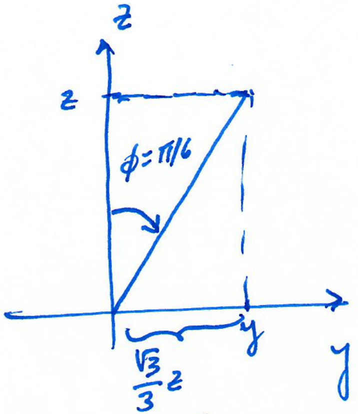
f2) Rotating $y = f(x)$ about the y-axis:

$$\begin{cases} x = v\cos u & \\ y = f(v) & a \le v \le b \\ z = v\sin u & 0 \le u \le 2\pi \end{cases}$$



Alternative for cone $\phi = \pi/6$.





$$y = (\tan\phi)\, z$$

$$y = \frac{\sqrt{3}}{3} z$$

$$\begin{cases} x = \frac{\sqrt{3}}{3} z \cos\theta, \\ y = \frac{\sqrt{3}}{3} z \sin\theta, \\ z = z \end{cases}$$

$0 \leq \theta \leq 2\pi$
$z \geq 0.$

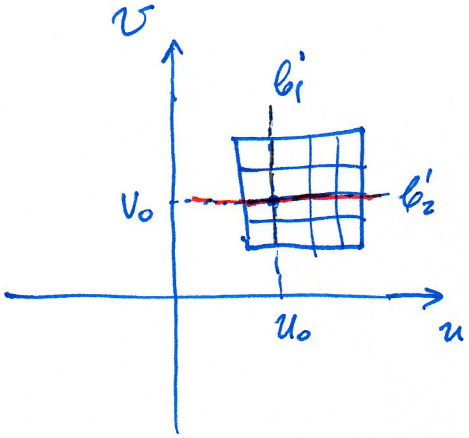
or in general for any ϕ, calling $u = \theta$, $v = z$.

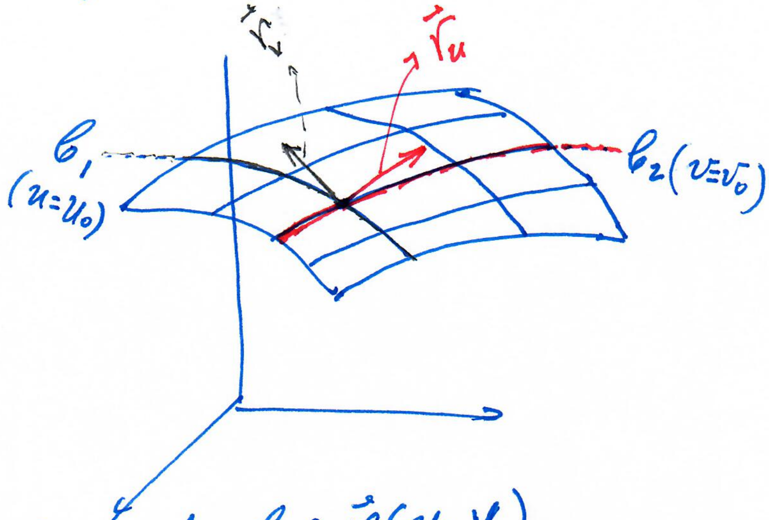
$$\begin{cases} x = \tan\phi\, v \cos u, \\ y = \tan\phi\, v \sin u, \\ z = v \end{cases}$$

$0 \leq u \leq 2\pi$
$z \geq 0$

Tangent Plane for Surfaces in parametric form.

$$\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$$





$\mathcal{C}_2 : \vec{r}(u,v_0)$ and $\mathcal{C}_1 : \vec{r}(u_0,v)$.

Therefore, <u>tang vector to $\mathcal{C}_2$ at (u_0,v_0)</u> is given by

$$\vec{r}_u(u_0,v_0) = \langle x_u(u_0,v_0),\ y_u(u_0,v_0),\ z_u(u_0,v_0) \rangle$$

Similarly, <u>tangent vector to $\mathcal{C}_1$ at (u_0,v_0)</u> is

$$\vec{r}_v(u_0,v_0) = \langle x_v(u_0,v_0),\ y_v(u_0,v_0),\ z_v(u_0,v_0) \rangle$$

Def. (Smooth Surfaces)

If $\vec{r}_u \times \vec{r}_v \neq 0$ at any point on S, then S is said to be smooth. (No corners).

Def For a Smooth Surface S given by the parametric equations

$$S: \ \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

$$(u,v) \in D \subseteq \mathbb{R}^2.$$

its tangent plane at $P_0(x_0,y_0,z_0)$, where $\vec{r}(u_0,v_0) = \langle x_0, y_0, z_0 \rangle$.

is the plane that contains the tangent vectors to the curve

$$\mathcal{C}_1 : \ \vec{r}(u_0,v) \quad \text{and} \quad \mathcal{C}_2 : \ \vec{r}(u,v_0)$$

$\vec{r}_u(u_0,v_0)$ and $\vec{r}_v(u_0,v_0)$, respectively.

Clearly, a normal vector to the tangent plane at P_0 is

$$\boxed{\vec{n}(x_0,y_0,z_0) = \vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0)}$$

Example: Find the tangent plane to the paraboloid

$$z = x^2 + y^2, \qquad 0 \le z \le 4$$

at the point $(1,1,2)$.

Answ: As we learned before

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$= f(1,1) + 2x\big|_{x=1}(x-1) + 2y\big|_{y=1}(y-1)$$

or $\quad z = 2 + 2(x-1) + 2(y-1) = 2x + 2y - 2$

or $\quad \boxed{2x + 2y - z = 2}$

Alternative using parametric representation

$$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2 \rangle, \qquad \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \end{array}$$

The point $(1,1,2)$ is obtained when $(r,\theta) = (\sqrt{2}, \pi/4)$

$r\cos\theta = 1, \; r\sin\theta = 1$

Then,

$$\vec{r}_r \times \vec{r}_\theta = \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \theta \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -2r^2\cos\theta, -2r^2\sin\theta, r\cos^2\theta + r\sin^2\theta \rangle$$

$$= r\langle -2r\cos\theta, -2r\sin\theta, 1 \rangle$$

Then,

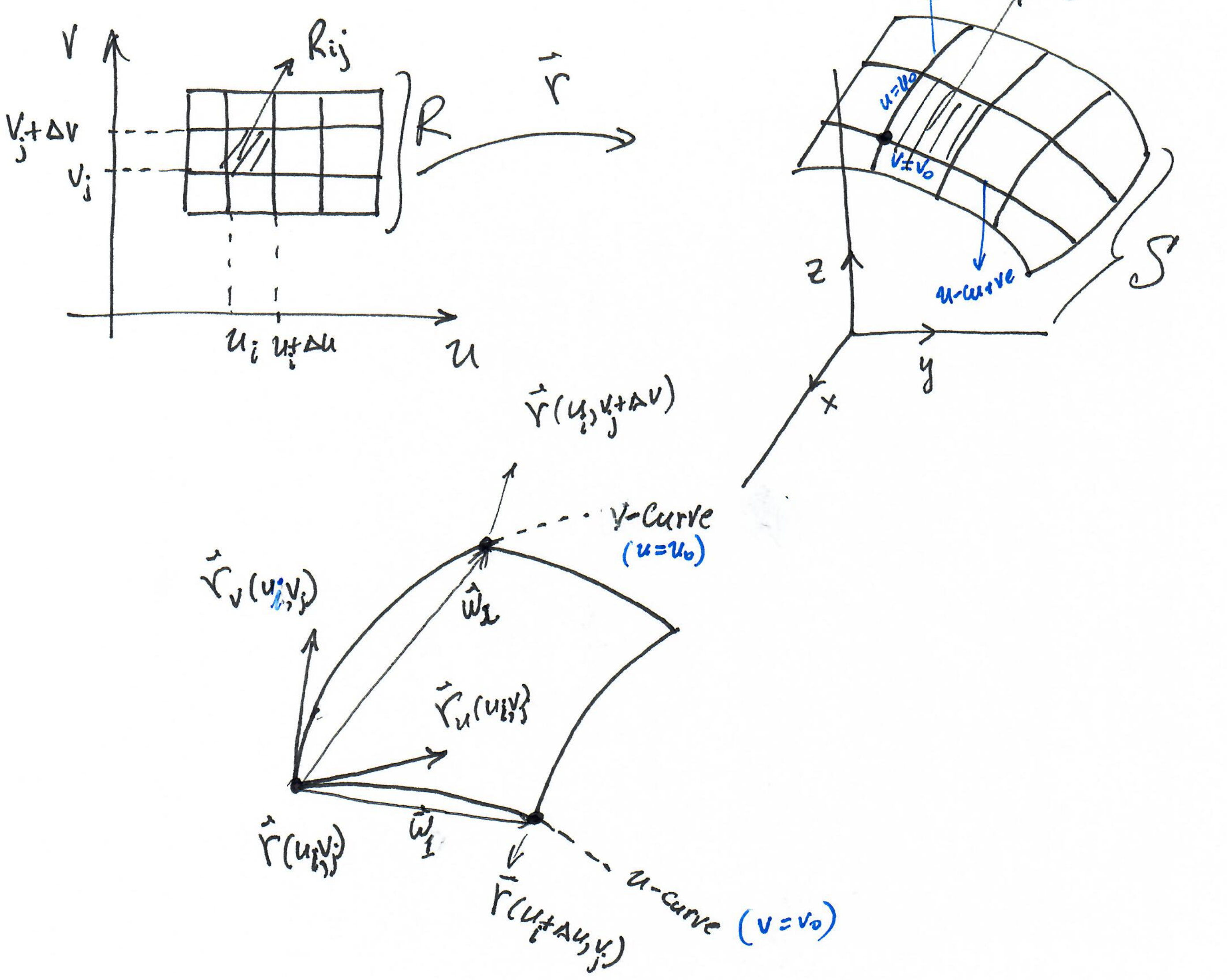
$$\vec{n}(\sqrt{2}, \pi/4) = \sqrt{2}\left\langle -2\sqrt{2}\,\frac{\sqrt{2}}{2}, -2\sqrt{2}\,\frac{\sqrt{2}}{2}, 1 \right\rangle$$

$$= \sqrt{2}\langle -2, -2, 1 \rangle \longrightarrow \langle 2, 2, -1 \rangle. \text{ OK} \checkmark$$

Then plane: $\quad 2(x-1) + 2(y-1) - (z-2) = 0$

or $\quad \boxed{2x + 2y - z = 2}$

Area of parametric Surfaces



The area of the surface cell S_{ij} can be approximated by the area of the parallelogram formed by $\vec{w}_1$ and $\vec{w}_2$ which is $\boxed{\|\vec{w}_1 \times \vec{w}_2\|}$

Now,

Mean Value Theorem

$$\underline{\vec{W}_1} = \vec{r}(u_i+\Delta u, v_j) - \vec{r}(u_i,v_j) = \vec{r}_u(u_i^*, v_j)\,\Delta u$$

$$\underline{\vec{W}_2} = \vec{r}(u_i, v_j+\Delta u) - \vec{r}(u_i,v_j) = \vec{r}_v(u_i, v_j^*)\,\Delta v$$

Therefore,

$$\boxed{\text{Area of } S_{ij} = \Delta S_{ij} \simeq \left\| \vec{r}_u(u_i^*, v_j) \times \vec{r}_v(u_i, v_j^*) \right\| \underset{\overset{\shortparallel}{\Delta R_{ij}}}{\Delta u \Delta v}}$$

Adding all these surface cells, the area of S can be approximated by

$$\text{Area of } S = \sum_{i,j} \Delta S_{ij} \simeq \sum_{i,j} \left\| \vec{r}_u(u_i^*, v_j) \times \vec{r}_v(u_i, v_j^*) \right\| \cdot \Delta u \Delta v$$

In the limit when $\Delta R_{ij} \to 0$, we obtain

$$u_i^* \to u_i$$
$$v_j^* \to v_j$$

$$\boxed{\begin{aligned} \text{Area of } S = \sigma &= \iint_R \left\| \vec{r}_u(u,v) \times \vec{r}_v(u,v) \right\| dA \\ &= \iint_R \left\| \vec{n}(u,v) \right\| dA \end{aligned}}$$

PS$_{11'}$

Find the Surface area of the cone

$$\boxed{z^2 = x^2 + y^2}, \qquad 0 \le z \le 1.$$

Answ: Using spherical coords, we can show $\phi = \pi/4$.

and

$$\vec{r}(r,\theta) = \begin{cases} x(\rho,\theta) = \rho \sin \pi/4 \cos\theta = \frac{\sqrt{2}}{2}\rho\cos\theta, \\ y(\rho,\theta) = \rho \sin \pi/4 \sin\theta = \frac{\sqrt{2}}{2}\rho\sin\theta, \qquad 0 \le \theta \le 2\pi \\ z(\rho,\theta) = \rho\cos\pi/4 = \frac{\sqrt{2}}{2}\rho, \qquad 0 \le z \le 1 \Rightarrow 0 \le \rho \le \frac{2}{\sqrt{2}} \end{cases}$$

Then

$$\vec{n} = \vec{r}_\rho \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sqrt{2}/2\cos\theta & \sqrt{2}/2\sin\theta & \sqrt{2}/2 \\ -\sqrt{2}/2\,\rho\sin\theta & \sqrt{2}/2\,\rho\cos\theta & 0 \end{vmatrix}$$

$$= \left\langle -\frac{1}{2}\rho\cos\theta, -\frac{1}{2}\rho\sin\theta, \frac{1}{2}\rho\cos^2\theta + \frac{1}{2}\rho\sin^2\theta \right\rangle$$

$$= \frac{1}{2}\rho\left\langle -\cos\theta, -\sin\theta, 1 \right\rangle$$

Thus,

$$S_{(\text{surface area})} = \frac{1}{2}\int_0^{2\pi}\int_0^{2/\sqrt{2}} \rho\sqrt{\cos^2\theta + \sin^2\theta + 1}\, d\rho\, d\theta$$

$$\theta \quad \rho$$

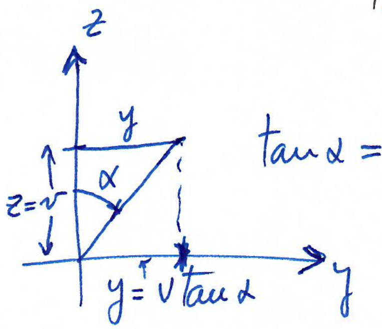
$$= \frac{\sqrt{2}}{2}(\cancel{2}\pi)\,\frac{\rho^2}{\cancel{2}}\Bigg|_0^{2/\sqrt{2}} = \cancel{\frac{\sqrt{2}}{2}}\,\pi\,\frac{\cancel{4}}{2} = \underline{\underline{\sqrt{2}\,\pi}}$$

BS 12

ALTERNATIVE 2

Area of Surface of cone

$$\boxed{z^2 = x^2 + y^2, \qquad 0 \le z \le 1}$$



$\tan\alpha = \frac{y}{v}$

$$\begin{cases} z = v \\ y = v\tan\alpha \sin u = \\ x = v\tan\alpha \cos u \end{cases}, \qquad \begin{matrix} 0 \le u \le 2\pi \\ 0 \le v \le 1 \end{matrix}$$

$$z^2 = v^2 \tan^2\alpha \Rightarrow \tan^2\alpha = 1 \Rightarrow \boxed{\alpha = \pi/4}$$

$$\therefore \boxed{\vec{r}(u,v) = \begin{cases} z = v \\ y = v\sin u \\ x = v\cos u \end{cases} \qquad \begin{matrix} 0 \le u \le 2\pi \\ 0 \le v \le 1 \end{matrix}}$$

$$\vec{r}_u = \langle -v\sin u, v\cos u, 0 \rangle$$

$$\vec{r}_v = \langle \cos u, \sin u, 1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 1 \end{vmatrix} = \langle v\cos u, v\sin u, -v(\underbrace{\sin^2 u + \cos^2 u}_{}\overset{=1}{}) \rangle$$

$$=$$

$$= \langle v\cos u, v\sin u, -v \rangle$$

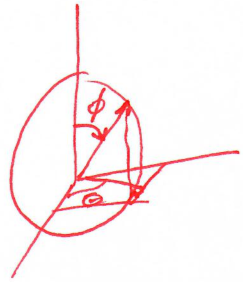
$$\text{Area of cone} = \int_{0\,(v)}^{1}\int_{0\,(u)}^{2\pi} |\vec{r}_u \times \vec{r}_v|\, du\, dv =$$

$$= \int_0^1\int_0^{2\pi} \left[v^2\cos^2 u + v^2\sin^2 u + v^2\right]^{1/2} du\, dv = \int_0^1\int_0^{2\pi} \sqrt{2}\, v\, du\, dv =$$

$$= \sqrt{2}\,\frac{v^2}{2}\Big|_0^1 = \cancel{2}\pi\sqrt{2}\,\frac{1}{\cancel{2}} = \sqrt{2}\,\pi \checkmark$$

Find Surface area of sphere of radius a.

$$x^2+y^2+z^2=a^2$$



or $\begin{cases} x = a\sin\phi\cos\theta \\ y = a\sin\phi\sin\theta \\ z = a\cos\phi \end{cases}$ $\quad 0 \le \overset{=u}{\phi} \le \pi$, $\quad 0 \le \overset{=v}{\theta} \le 2\pi$

Using the previous def.

$$A(S) = \int_{\underset{\theta}{0}}^{2\pi} \int_{\underset{\phi}{0}}^{\pi} |r_\phi \times r_\theta| \, d\phi \, d\theta$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_\phi & y_\phi & z_\phi \\ x_\theta & y_\theta & z_\theta \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \hat{i}\, a^2\sin^2\phi\cos\theta + \hat{j}\, a^2\sin^2\phi\sin\theta + \hat{k}\left(\underbrace{a^2\cos^2\theta\cos\phi\sin\phi + a^2\sin^2\theta\cos\phi\sin\phi}_{a^2\cos\phi\sin\phi}\right)$$

$$\Rightarrow |r_\phi \times r_\theta| = \sqrt{a^4\left(\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \cos^2\phi\sin^2\phi\right)} =$$

$$= a^2\sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi} = a^2\sqrt{\sin^2\phi\,(\sin^2\phi + \cos^2\phi)} =$$

$$= a^2\sin\phi.$$

$$\therefore A(S) = \int_0^{2\pi}\int_0^{\pi} a^2\sin\phi \, d\phi\, d\theta = a^2\int_0^{2\pi} -\cos\phi\Big|_0^{\pi} d\theta = a^2\int_0^{2\pi} 2\, d\theta = \underline{\underline{4\pi a^2}}$$

In the particular case of a surface S defined by a function f of two variables,

$$S: \quad z = f(x,y), \quad (x,y) \in R$$

which is conts. diff.

We can use the parametrization

$$\vec{r}(u,v) = \langle u, v, f(u,v) \rangle, \quad (u,v) \in R.$$

Then, $\vec{n}(u,v) = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix}$

$$= \langle -f_u, -f_v, 1 \rangle$$

So

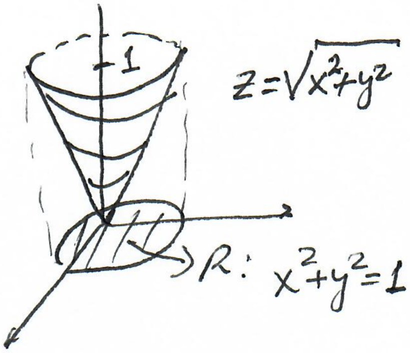
$$\text{Area of } S = \sigma = \iint_R \sqrt{f_u^2 + f_v^2 + 1} \, du\,dv$$

or

$$\text{Area of } S = \sigma = \iint_R \sqrt{f_x^2 + f_y^2 + 1} \, dx\,dy.$$

Surface of a Cone (Revisited)

$$z^2 = x^2 + y^2 , \qquad 0 \leq z \leq 1$$



$$\text{Area of } S = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dx dy$$

$$z = f(x,y) = \sqrt{x^2+y^2} \Rightarrow f_x = \frac{x}{\sqrt{x^2+y^2}} , \quad f_y = \frac{y}{\sqrt{x^2+y^2}}$$

Then

$$\text{Area of } S = \iint_{\substack{R \\ x^2+y^2=1}} \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}\, dx dy$$

$$= \iint_R \sqrt{2}\, dx dy = \sqrt{2}\, \pi (1)^2 = \underline{\sqrt{2}\, \pi}$$

If $z = f(x,y)$, $(x,y) \in D$ represent a surface S

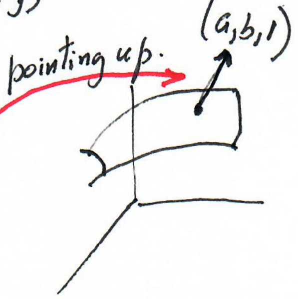
then a possible parametrization: $x=x$, $y=y$, $z=f(x,y)$

Thus,

$$\hat{r}_x = \langle 1, 0, f_x \rangle , \quad \hat{r}_y = \langle 0, 1, f_y \rangle$$

and

$$\vec{n} = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \langle -f_x, -f_y, 1 \rangle$$

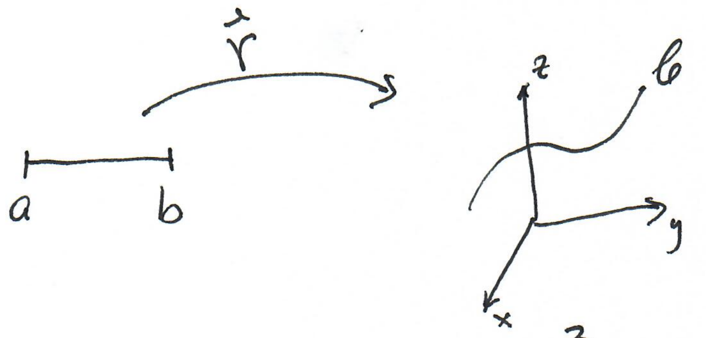


16.6 Parametric Surfaces and Their Areas.

Curves in 3-D are represented in parametric form as

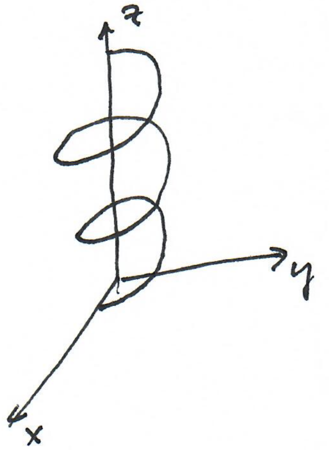
$$\mathcal{C}: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle \quad , \quad a \le t \le b$$

or $x = x(t)$, $y = y(t)$, $z = z(t)$



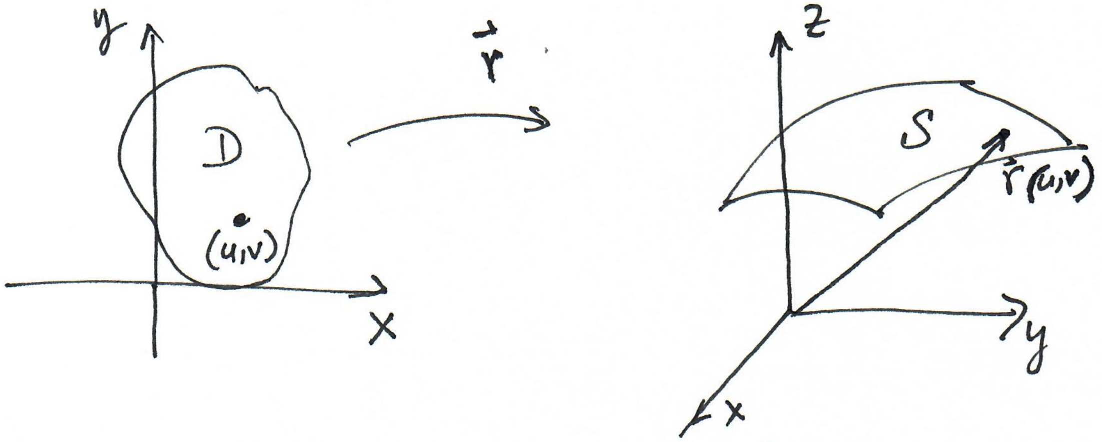
$$\vec{r} : [a,b] \longrightarrow \mathbb{R}^3$$

Example: $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 3\pi$



Similarly, Surfaces can be represented parametrically by

$$\vec{r} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$



$$\vec{r}(u,v) = \left\langle x(u,v), y(u,v), z(u,v) \right\rangle, \quad (u,v) \in D \subseteq \mathbb{R}^2$$

16.6

PS3

Parametric Surfaces and Their Areas.

Example 1.- Cylinder

$x^2+y^2=9$,

$0 \le z \le 10$

Parametrization:

$$\left.\begin{array}{l} x = x(\theta,z) = 3\cos\theta \\ y = y(\theta,z) = 3\sin\theta \\ z = z(\theta,z) = z \end{array}\right\} \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le z \le 10 \end{array}$$

or $\vec{r}(\theta,z) = \langle 3\cos\theta, 3\sin\theta, z\rangle$

Example 2.- Sphere

$x^2+y^2+z^2=a^2$

$$\left\{\begin{array}{l} x = x(\phi,\theta) = a\sin\phi\cos\theta, \\ y = y(\phi,\theta) = a\sin\phi\sin\theta, \\ z = z(\phi,\theta) = a\cos\phi, \end{array}\right. \quad \begin{array}{l} 0 \le \phi \le \pi \\ 0 \le \theta \le 2\pi \end{array}$$

If $u=\phi$ and $v=\theta$

$\Rightarrow \vec{r}(u,v) = \langle a\sin u\cos v, a\sin u\sin v, a\cos u\rangle$ $\quad 0 \le u \le \pi$, $0 \le v \le 2\pi$

Example 3.- Paraboloid : $z = x^2+y^2$, $0 \le z \le 4$

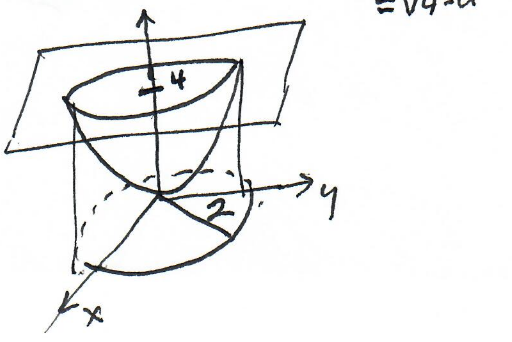
I) $\vec{r}(x,y) = \langle x, y, x^2+y^2\rangle$ or $\vec{r}(u,v) = \langle u, v, u^2+v^2\rangle$

$-2 \le u \le 2$, $-\sqrt{4-u^2} \le v \le \sqrt{4-u^2}$

or II) $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2\rangle$

$0 \le \theta \le 2\pi$

$0 \le r \le 2$



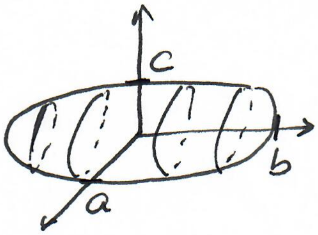
Example 4 $\quad r(u,v) = \langle (2+\sin v)\cos u, (2+\sin v)\sin u, u+\cos v \rangle$

$0 \le u \le 4\pi$
$0 \le v \le 2\pi$

(Example 2 in book 7E).

Example 5 Ellipsoid with Semi-axes a, b, and c

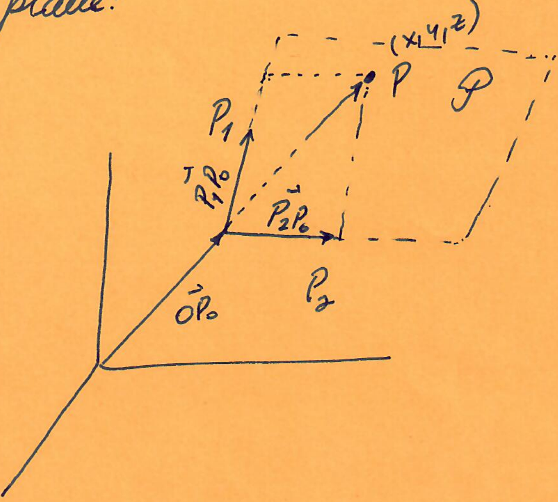
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



$$S': \begin{cases} x = a\cos u \sin v \\ y = b \sin u \sin v \\ z = c \cos v \end{cases} \qquad \begin{matrix} u \in [0, 2\pi] \\ v \in [0, \pi] \end{matrix}$$

b2) Another parametrization of the plane:

$$\boxed{\mathcal{P}: \quad 3x + 2y - 2z = 5}$$



Consider the three points:

$P_0(1,1,0)$, $P_1(3,-1,1)$, $P_2(5,-1,4)$

on the plane $\mathcal{P}$

and form the vectors:

$$\vec{P_1P_0} = \vec{\omega}_1 = (3,-1,1) - (1,1,0) = (2,-2,1)$$

$$\vec{P_2P_0} = \vec{\omega}_2 = (5,-1,4) - (1,1,0) = (4,-2,4)$$

then any point $P(x,y,z)$ in $\mathcal{P}$ satisfies

$$\vec{PP_0} = u\,\vec{\omega}_1 + v\,\vec{\omega}_2 + \vec{OP_0}$$

$$= u\,(2,-2,1) + v\,(4,-2,4) + (1,1,0).$$

$$= (2u + 4v + 1, \; -2u - 2v + 1, \; u + 4v)$$

or parametric representation of plane $\mathcal{P}$.

$$\begin{cases} x = x(u,v) = 2u + 4v + 1 \\ y = y(u,v) = -2u - 2v + 1. \\ z = z(u,v) = u + 4v \end{cases}$$

Parametric Representation of cone: (Half-cone)

$$z^2 = 3(x^2+y^2), \qquad z \geq 0$$

Using spherical coordinates

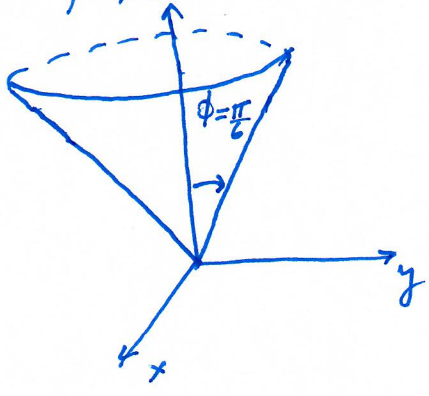
$$\begin{cases} x = \rho \sin\phi \cos\theta, & 0 \le \theta \le 2\pi \\ y = \rho \sin\phi \sin\theta, & 0 \le \phi \le \pi \\ z = \rho \cos\phi \end{cases}$$

$$\rho^2 \cos^2\phi = 3\rho^2 \sin^2\phi \Rightarrow \tan^2\phi = \frac{1}{3} \Rightarrow \tan\phi = \frac{\sqrt{3}}{3}$$

$$\Rightarrow \begin{matrix} \sin\phi = 1/2 \\ \cos\phi = \sqrt{3}/2 \end{matrix} \Rightarrow \phi = \frac{\pi}{6}$$

Then, a natural parametrization is

$$\begin{cases} x = \rho \sin\pi/6 \cos\theta = \frac{\rho}{2}\cos\theta, & 0 \le \theta \le \pi \\ y = \rho \sin\pi/6 \sin\theta = \frac{\rho}{2}\sin\theta, & \rho \geq 0 \\ z = \rho \cos\pi/6 = \frac{\sqrt{3}}{2}\rho \end{cases}$$



If we call $u = \rho$ and $v = \theta$

Then,

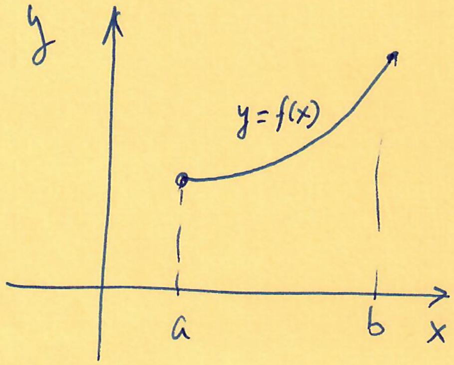
$$\begin{cases} x = \frac{1}{2} u \cos v, & u \geq 0 \\ y = \frac{1}{2} u \sin v, & 0 \le v \le 2\pi. \\ z = \frac{\sqrt{3}}{2} u \end{cases}$$

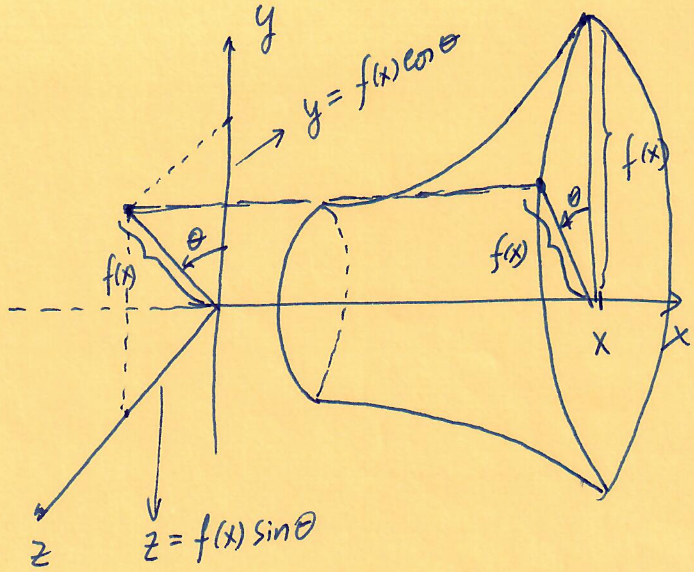
PS_x

f) Surfaces of revolution

Consider the function $y = f(x)$, $a \le x \le b$.

And rotate it about the x-axis.

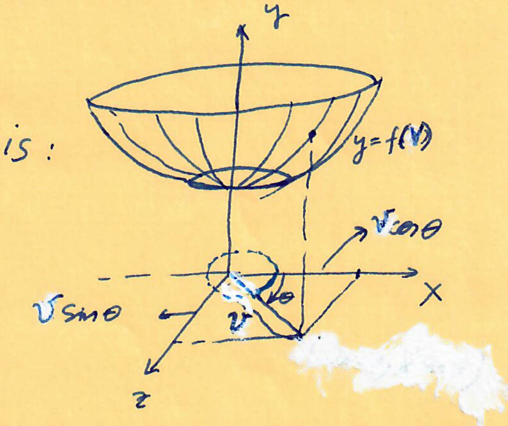




Parametric equations: $\theta = v$

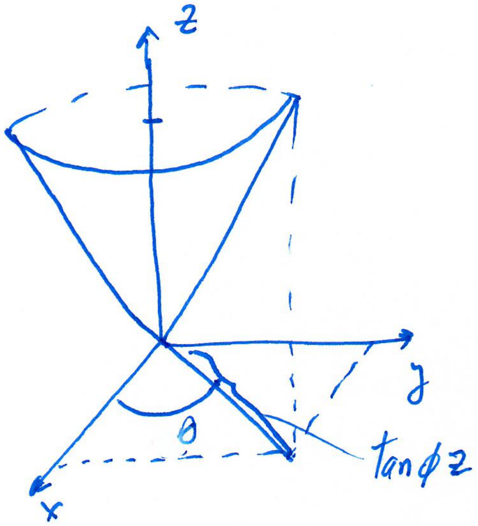
$$\begin{cases} x = u & a \le u \le b \\ y = f(u) \cos v & 0 \le v \le 2\pi. \\ z = f(u) \sin v & \end{cases}$$

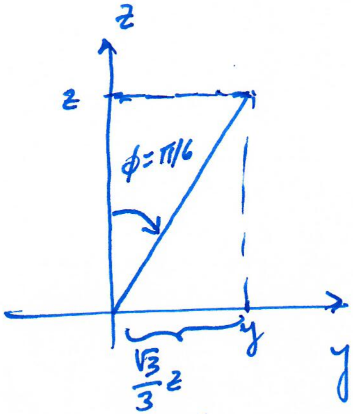
f2) Rotating $y = f(x)$ about the y-axis:



$$\begin{cases} x = v \cos u & \\ y = f(v) & a \le v \le b \\ z = v \sin u & 0 \le u \le 2\pi \end{cases}$$

Alternative for Cone $\phi = \pi/6$.





$$y = (\tan\phi)\, z$$

$$y = \frac{\sqrt{3}}{3} z$$

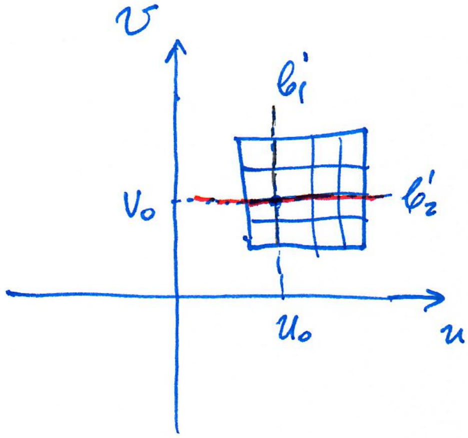
$$\begin{cases} x = \frac{\sqrt{3}}{3} z \cos\theta, \\ y = \frac{\sqrt{3}}{3} z \sin\theta, \\ z = z \end{cases} \qquad \begin{matrix} 0 \le \theta \le 2\pi \\ z \geq 0. \end{matrix}$$

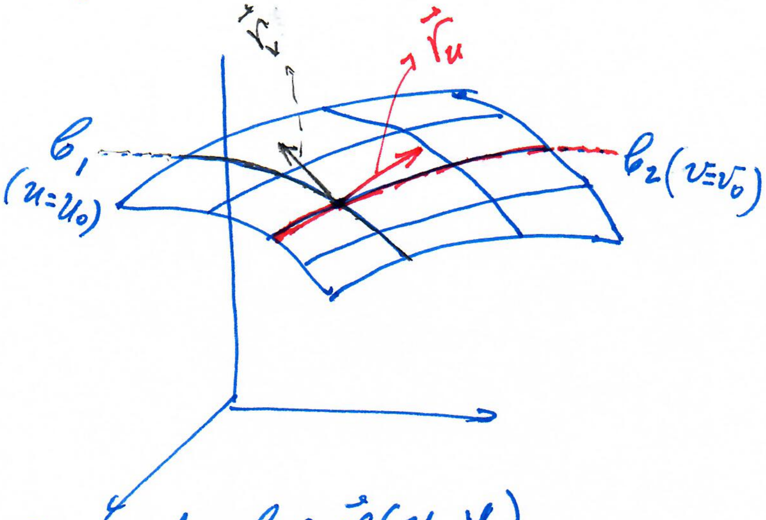
or in general for any ϕ, calling $u = \theta$, $v = z$.

$$\begin{cases} x = \tan\phi\, v \cos u, \\ y = \tan\phi\, v \sin u, \\ z = v \end{cases} \qquad \begin{matrix} 0 \le u \le 2\pi \\ z \geq 0 \end{matrix}$$

Tangent Plane for Surfaces in parametric form.

$$\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$$





$\mathcal{C}_2 : \vec{r}(u,v_0)$ and $\mathcal{C}_1 : \vec{r}(u_0,v)$.

Therefore, tang vector to $\mathcal{C}_2$ at (u_0,v_0) is given by

$$\vec{r}_u(u_0,v_0) = \langle x_u(u_0,v_0),\ y_u(u_0,v_0),\ z_u(u_0,v_0) \rangle$$

Similarly, tangent vector to $\mathcal{C}_1$ at (u_0,v_0) is

$$\vec{r}_v(u_0,v_0) = \langle x_v(u_0,v_0),\ y_v(u_0,v_0),\ z_v(u_0,v_0) \rangle$$

Def. (Smooth Surfaces)

If $\vec{r}_u \times \vec{r}_v \neq 0$ at any point on S, then S is said to be Smooth. (No corners).

Def: For a Smooth Surface S given by the parametric equations

$$S: \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

$$(u,v) \in D \subseteq \mathbb{R}^2.$$

its tangent plane at $P_0(x_0, y_0, z_0)$, where $\vec{r}(u_0,v_0) = \langle x_0, y_0, z_0 \rangle$.

is the plane that contains the tangent vectors to the curve

$$\mathcal{C}_1: \vec{r}(u_0, v) \quad \text{and} \quad \mathcal{C}_2: \vec{r}(u, v_0)$$

$\vec{r}_u(u_0,v_0)$ and $\vec{r}_v(u_0,v_0)$, respectively.

Clearly, a normal vector to the tangent plane at P_0 is

$$\boxed{\vec{n}(x_0,y_0,z_0) = \vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0)}$$

Example: Find the tangent plane to the paraboloid

$$z = x^2 + y^2, \qquad 0 \le z \le 4$$

at the point $(1,1,2)$.

Answ: As we learned before

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$= f(1,1) + 2x\big|_{x=1}(x-1) + 2y\big|_{y=1}(y-1)$$

or $$z = 2 + 2(x-1) + 2(y-1) = 2x + 2y - 2$$

or $$\boxed{2x + 2y - z = 2}$$

Alternative using parametric representation

$$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2\rangle, \qquad \begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \end{matrix}$$

The point $(1,1,2)$ is obtained when $(r,\theta) = (\sqrt{2}, \pi/4)$

$r\cos\theta = 1, \ r\sin\theta = 1$

Then,

$$\vec{r}_r \times \vec{r}_\theta = \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \theta \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -2r^2\cos\theta, -2r^2\sin\theta, r\cos^2\theta + r\sin^2\theta \rangle$$

$$= r\langle -2r\cos\theta, -2r\sin\theta, 1\rangle$$

Then,

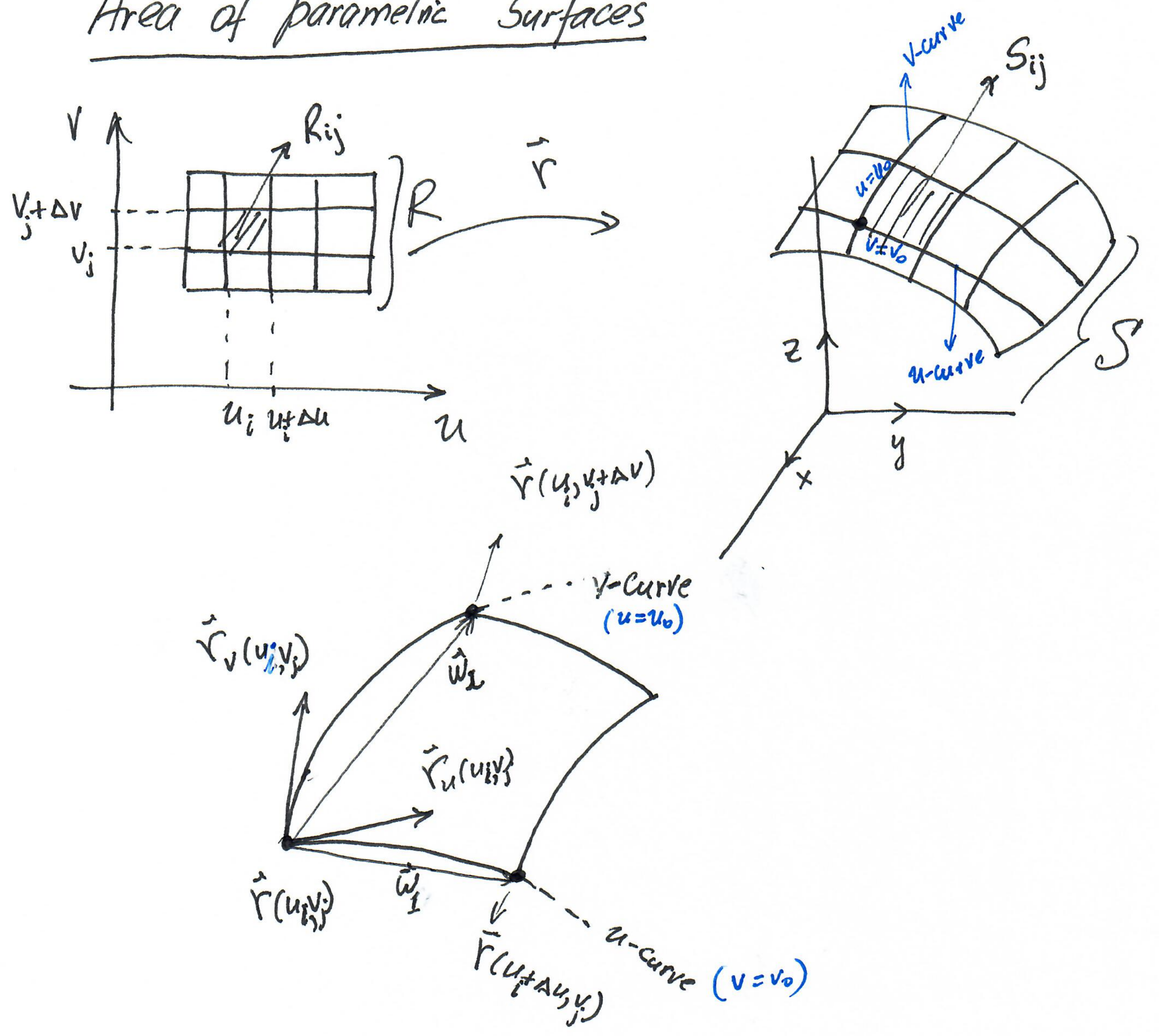
$$\vec{n}(\sqrt{2}, \pi/4) = \sqrt{2}\left\langle -2\sqrt{2}\,\frac{\sqrt{2}}{2}, -2\sqrt{2}\,\frac{\sqrt{2}}{2}, 1\right\rangle$$

$$= \sqrt{2}\langle -2, -2, 1\rangle \longrightarrow \langle 2, 2, -1\rangle. \text{ OK} \checkmark$$

Then plane: $2(x-1) + 2(y-1) - (z-2) = 0$

or $$\boxed{2x + 2y - z = 2}$$

Area of parametric Surfaces



The area of the surface cell S_{ij} can be approximated by the area of the parallelogram formed by $\vec{w}_1$ and $\vec{w}_2$ which is $\boxed{\|\vec{w}_1 \times \vec{w}_2\|}$

Now,

Mean Value Theorem

$$\underline{\underline{\vec{w}_1}} = \vec{r}(u_i+\Delta u, v_j) - \vec{r}(u_i, v_j) = \vec{r}_u(u_i^*, v_j)\,\Delta u$$

$$\underline{\underline{\vec{w}_2}} = \vec{r}(u_i, v_j+\Delta u) - \vec{r}(u_i, v_j) = \vec{r}_v(u_i, v_j^*)\,\Delta v$$

Therefore,

$$\boxed{\text{Area of } S_{ij} = \Delta S_{ij} \simeq \left\| \vec{r}_u(u_i^*, v_j) \times \vec{r}_v(u_i, v_j^*) \right\| \underset{\Delta R_{ij}}{\underset{\shortparallel}{\Delta u \Delta v}}}$$

Adding all these surface cells, the area of S can be approximated by

$$\text{Area of } S = \sum_{i,j} \Delta S_{ij} \simeq \sum_{i,j} \left\| \vec{r}_u(u_i^*, v_j) \times \vec{r}_v(u_i, v_j^*) \right\| \cdot \Delta u \Delta v$$

In the limit when $\Delta R_{ij} \to 0$, we obtain

$$u_i^* \to u_i$$
$$v_j^* \to v_j$$

$$\boxed{\begin{aligned}\text{Area of } S = \sigma &= \iint_R \left\| \vec{r}_u(u,v) \times \vec{r}_v(u,v) \right\| dA \\ &= \iint_R \left\| \vec{n}(u,v) \right\| dA\end{aligned}}$$

Find the Surface area of the cone

$$\boxed{z^2 = x^2 + y^2,} \qquad 0 \le z \le 1.$$

Answ: Using spherical coords, we can show $\phi = \pi/4$.

and

$$\vec{r}(r,\theta) = \begin{cases} x(\rho,\theta) = \rho \sin \pi/4 \cos\theta = \frac{\sqrt{2}}{2}\rho\cos\theta, & \\ y(\rho,\theta) = \rho \sin\pi/4 \sin\theta = \frac{\sqrt{2}}{2}\rho\sin\theta, & 0 \le \theta \le 2\pi \\ z(\rho,\theta) = \rho\cos\pi/4 = \frac{\sqrt{2}}{2}\rho, & 0 \le z \le 1 \Rightarrow 0 \le \rho \le \frac{2}{\sqrt{2}} \end{cases}$$

Then

$$\vec{n} = \vec{r}_\rho \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sqrt{2}/2 \cos\theta & \sqrt{2}/2 \sin\theta & \sqrt{2}/2 \\ -\sqrt{2}/2\, \rho\sin\theta & \sqrt{2}/2\, \rho\cos\theta & 0 \end{vmatrix}$$

$$= \left\langle -\frac{1}{2}\rho\cos\theta, -\frac{1}{2}\rho\sin\theta, \frac{1}{2}\rho\cos^2\theta + \frac{1}{2}\rho\sin^2\theta \right\rangle$$

$$= \frac{1}{2}\rho \left\langle -\cos\theta, -\sin\theta, 1 \right\rangle$$

Thus,

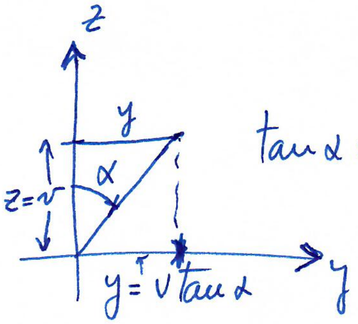
$$S_{(\text{surface area})} = \frac{1}{2}\int_{\substack{0\\ \theta}}^{2\pi}\int_{\substack{0\\ \rho}}^{2/\sqrt{2}} \rho\sqrt{\cos^2\theta + \sin^2\theta + 1}\, d\rho\, d\theta$$

$$= \frac{\sqrt{2}}{2}(\cancel{2}\pi)\,\frac{\rho^2}{\cancel{2}}\Bigg|_0^{2/\sqrt{2}} = \frac{\sqrt{2}}{\cancel{2}}\pi\frac{\cancel{4}}{2} = \underline{\underline{\sqrt{2}\,\pi}}$$

ALTERNATIVE 2

Area of Surface of cone

$$\boxed{z^2 = x^2 + y^2, \qquad 0 \le z \le 1}$$



$\tan\alpha = \frac{y}{V}$

$$\begin{cases} z = v \\ y = v\tan\alpha \sin u = \\ x = v\tan\alpha \cos u \end{cases}, \qquad \begin{array}{l} 0 \le u \le 2\pi \\ 0 \le v \le 1 \end{array}$$

$$z^2 = v^2 \tan^2\alpha \Rightarrow \tan^2\alpha = 1 \Rightarrow \boxed{\alpha = \pi/4}$$

$$\therefore \boxed{\vec{r}(u,v) = \begin{cases} z = v \\ y = v\sin u \\ x = v\cos u \end{cases} \qquad \begin{array}{l} 0 \le u \le 2\pi \\ 0 \le v \le 1 \end{array}}$$

$$\vec{r}_u = \langle -v\sin u, v\cos u, 0 \rangle$$

$$\vec{r}_v = \langle \cos u, \sin u, 1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 1 \end{vmatrix} = \langle v\cos u, v\sin u, -v(\overset{=1}{\sin^2 u + \cos^2 u}) \rangle$$

$$=$$

$$= \langle v\cos u, v\sin u, -v \rangle$$

Area of cone $= \displaystyle\int_{\underset{v}{0}}^{1} \int_{\underset{u}{0}}^{2\pi} |\vec{r}_u \times \vec{r}_v| \, du\, dv =$

$$= \int_0^1 \int_0^{2\pi} [v^2\cos^2 u + v^2 \sin^2 u + v^2]^{1/2} \, du\, dv = \int_0^1 \int_0^{2\pi} \sqrt{2}\, v \, du\, dv =$$

$$= \sqrt{2}\,2\pi \left.\frac{v^2}{2}\right|_0^1 = \cancel{2}\pi\sqrt{2}\,\frac{1}{\cancel{2}} = \sqrt{2}\,\pi \quad ✓$$

PS 12'

Find Surface area of paraboloid:

$$z = x^2 + y^2, \qquad 0 \le z \le 4.$$

answ:

Parametric Equation:

As we already found out $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2 \rangle$

$0 \le \theta \le 2\pi$

$0 \le r \le 2$

$$\vec{n} = \vec{r}_r \times \vec{r}_\theta =$$

$$= r \langle -2r\cos\theta, -2r\sin\theta, 1 \rangle$$

Then,

$$S_{(\text{Surface area})} = \int_{\underset{\theta}{0}}^{2\pi} \int_{\underset{r}{0}}^{2} \sqrt{[4r^2(\cos^2\theta + \sin^2\theta) + 1]\, r^2}\, dr\, d\theta$$

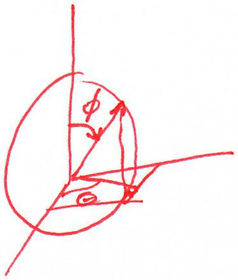
$$= \int_0^{2\pi} \int_{\underset{r}{0}}^{2} \sqrt{4r^2+1}\; r\, dr\, d\theta$$

$$= 2\pi \int_0^2 (4r^2+1)^{1/2} r\, dr = 2\pi \left(\frac{1}{8}\right) \int_0^2 (4r^2+1)^{1/2} 8r\, dr$$

$$= \frac{\pi}{4} \left.(4r^2+1)^{3/2} \frac{2}{3}\right|_0^2 = \frac{2\pi}{3\,4}\left[(17)^{3/2} - 1\right] = \frac{\pi}{6}\left(17\sqrt{17} - 1\right)$$

Find Surface area of sphere of radius a.

$$\underline{x^2+y^2+z^2=a^2}$$



or $\begin{cases} x = a\sin\phi\cos\theta \\ y = a\sin\phi\sin\theta \\ z = a\cos\phi \end{cases}$ $\quad 0 \le \overset{=u}{\phi} \le \pi$, $\quad 0 \le \overset{=v}{\theta} \le 2\pi$

Using the previous def.

$$A(S) = \int_{\underset{\theta}{0}}^{2\pi}\int_{\underset{\phi}{0}}^{\pi} |r_\phi \times r_\theta| \, d\phi \, d\theta$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_\phi & y_\phi & z_\phi \\ x_\theta & y_\theta & z_\theta \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \hat{i}\, a^2\sin^2\phi\cos\theta + \hat{j}\, a^2\sin^2\phi\sin\theta + \hat{k}\left(\underbrace{a^2\cos^2\theta\cos\phi\sin\phi + a^2\sin^2\theta\cos\phi\sin\phi}_{a^2\cos\phi\sin\phi}\right)$$

$$\Rightarrow |r_\phi \times r_\theta| = \sqrt{a^4\left(\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \cos^2\phi\sin^2\phi\right)} =$$

$$= a^2\sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi} = a^2\sqrt{\sin^2\phi\left(\sin^2\phi+\cos^2\phi\right)} =$$

$$= a^2\sin\phi.$$

$$\therefore A(S) = \int_0^{2\pi}\int_0^{\pi} a^2\sin\phi \, d\phi \, d\theta = a^2\int_0^{2\pi} -\cos\phi\Big|_0^{\pi} d\theta = a^2\int_0^{2\pi} 2\, d\theta = \underline{\underline{4\pi a^2}}$$

In the particular case of a surface S defined by a function f of two variables,

$$S: \quad z = f(x,y), \quad (x,y) \in R$$

which is conts. diff.

We can use the parametrization

$$\vec{r}(u,v) = \langle u, v, f(u,v) \rangle, \quad (u,v) \in R.$$

Then, $\vec{n}(u,v) = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix}$

$$= \langle -f_u, -f_v, 1 \rangle$$

So

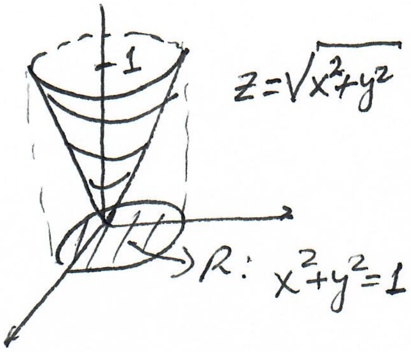
$$\text{Area of } S = \sigma = \iint_R \sqrt{f_u^2 + f_v^2 + 1}\, du\, dv$$

or

$$\text{Area of } S = \sigma = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy.$$

Surface of a Cone (Revisited)

$$z^2 = x^2 + y^2, \qquad 0 \le z \le 1$$



$$\text{Area of } S = \iint_R \sqrt{1+f_x^2+f_y^2}\,dx\,dy$$

$$z=f(x,y)=\sqrt{x^2+y^2} \Rightarrow f_x = \frac{x}{\sqrt{x^2+y^2}}, \quad f_y = \frac{y}{\sqrt{x^2+y^2}}$$

Then

$$\text{Area of } S = \iint_{\substack{R\\x^2+y^2=1}} \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\,dx\,dy$$

$$= \iint_R \sqrt{2}\,dx\,dy = \sqrt{2}\,\pi(1)^2 = \underline{\sqrt{2}\,\pi}$$

If $z=f(x,y)$, $(x,y)\in D$ represent a surface S

then a possible parametrization: $x=x,\ y=y,\ z=f(x,y)$

Thus,

$$\vec{r}_x = \langle 1,0,f_x\rangle, \quad \vec{r}_y = \langle 0,1,f_y\rangle$$

and

$$\vec{n} = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \langle -f_x, -f_y, 1\rangle$$